An existence result for a quasilinear system with gradient term under the Keller-Osserman conditions

Dragos-Patru Covei

Constantin Brancusi University of Tg-Jiu, Calea Eroilor Nr.30, Targu-Jiu, Gorj, Romania

E-mail: patrucovei@yahoo.com

Abstract

We deal with existence of entire solutions for the quasilinear elliptic system of this type $\Delta_p u_i + h_i(|x|)|\nabla u_i|^{p-1} = a_i(|x|) f_i(u_1, u_2)$ on $\mathbb{R}^N$ $(N \geq 3,\ i = 1, 2)$ where $N-1 \geq p > 1$, Δ_p is the p-Laplacian operator and h_i, a_i, f_i are suitable functions. The results of this paper supplement the existing results in the literature and improve those obtained by Xinguang Zhang and Lishan Liu, The existence and nonexistence of entire positive solutions of semilinear elliptic systems with gradient term, Journal of Mathematical Analysis and Applications, Volume 371, Issue 1, 1 November 2010, Pages 300-308).

2000 Mathematics Subject Classification: 35J60;35J62;35J66;35J92;58J10;58J20.

Key words: Entire solution; Large solution; Elliptic system.

1 Introduction

In the present paper we establish a new result concerning the existence of solutions for the quasilinear elliptic system

$$\begin{cases} \Delta_p u_1(r) + h_1(r)|\nabla u_1(r)|^{p-1} = a_1(r) f_1(u_1(r), u_2(r)) \ , \\ \Delta_p u_2(r) + h_2(r)|\nabla u_2(r)|^{p-1} = a_2(r) f_2(u_1(r), u_2(r)) \ , \end{cases} \tag{1.1}$$

where $r := |x|$ for $x \in \mathbb{R}^N$ $(N-1 \geq p > 1)$ is the Euclidean norm, Δ_p is the so called p-Laplace operator defined by $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$. It will be assumed throughout this paper that a_j, h_j $(j = 1, 2)$ are nonnegative nontrivial $C(\mathbb{R}^N)$ functions, while $f_j : [0, \infty)^2 \to [0, \infty)$ $(j = 1, 2)$ are continuous and nondecreasing functions in each variable and verify $f_j(s_1, s_2) > 0$ whenever $s_i > 0$ for some $i = 1, 2$ together with the "Keller-Osserman type" condition

$$I(\infty) := \lim_{r \to \infty} I(r) = \infty \tag{1.2}$$

where $I(r) := \int_a^r [F(s)]^{-1/2} ds$ for $r \geq a > 0$, $F(s) := \int_0^s \sum_{i=1}^{2} f_i(t, t)\, dt$.

For a single equation of the form $\Delta u = f(u)$ where $f(u)$ is positive, real continuous function defined for all real u and nondecreasing the existence of entire large solutions is equivalent to a condition on f known as the Keller–Osserman condition

$$\int_{u_0}^{\infty} \left(\int_0^t f(s)\, ds\right)^{-1/2} dt = \infty \text{ for } \ u_0 > 0, \tag{1.3}$$

(see [4], [10]). In particular, Keller and Osserman prove that a necessary and sufficient condition for the considered problem to have an entire solution is that f satisfies (1.3). Such a solution will necessarily satisfies $\lim_{|x|\to\infty} u(x) = \infty$ and hence be a large solution. Moreover, Keller applied the results to electrohydrodynamics, namely to the problem of the equilibrium of a charged gas in a conducting container, see [5].

There is by now a broad literature regarding the study of solutions for (1.1). Basic results in the study of solutions for such problems have been obtained in the last few decades in the works of [1, 3, 6, 7, 8, 9, 12] and their references. We comment below on a few further results closer to our interests in the present article.

Regarding (1.1), Zhang and Liu [12] studied the existence of entire large positive solutions of the system

$$\begin{cases} \Delta u_1 + |\nabla u_1| = a_1(r) f_1(u_1, u_2), \\ \Delta u_2 + |\nabla u_2| = a_2(r) f_2(u_1, u_2). \end{cases}$$

In [12], the authors imposed on a_1, a_2, f_1 and f_2 satisfying the above conditions and instead of the Keller-Osserman condition the following

$$\int_a^\infty \frac{ds}{f_1(s,s) + f_2(s,s)} = \infty \text{ for } r \geq a > 0. \tag{1.4}$$

Obviously, (1.4) implies (1.2).

Finally, we note that the study of large solutions for (1.1) when the integral in (1.2) is finite has been the subject of the article [11].

Motivated by papers [11] and [12] we are interested in another type of nonlinearity f_i $(i = 1, 2)$ in order to obtain the existence of entire large/bounded positive solutions of (1.1).

The main reult of this article is:

Theorem 1.1. *Under the above hypotheses there are infinitely many positive entire radial solutions of system (1.1). Suppose furthermore that* $r^{\frac{p(N-1)}{p-1}} \sum_{j=1}^{2} e^{\frac{p}{p-1}\int_0^r h_j(t)dt} a_j(r)$ *is nondecreasing for large* r. *Then the solutions:*

i) are bounded if there exists a positive number ε *such that*

$$\int_0^\infty t^{1+\varepsilon} \left(\sum_{j=1}^{2} e^{\frac{p}{p-1}\int_0^t h_j(t)dt} a_j(t) \right)^{2/p} dt < \infty, \tag{1.5}$$

ii) are large if

$$\int_0^\infty \left(\frac{e^{-\int_0^t h_j(s)ds}}{t^{N-1}} \int_0^t s^{N-1} e^{\int_0^s h_j(t)dt} a_j(s)\, ds \right)^{1/(p-1)} dt = \infty \text{ for all } j = 1, 2 \tag{1.6}$$

hold.

Our main result are new, because no solutions have been detected yet for the system of the form (1.1) under the Keller-Osserman conditions (1.2). We mention that we can prove similar results for f_1 and f_2 being non-monotonic as in [4], [11]. Since in this case the proof is as for the monotone case we omit it.

2 Proof of the Theorem 1.1

We start by showing that (1.1) has positive radial solutions. The proof is inspired by [3] with some new ideas. Note that radial solutions of (1.1) are radial solutions of the system

$$\begin{cases} (p-1)\, u_1'(r)^{p-2}\, u_1'' + \frac{N-1}{r} u_1'(r)^{p-1} + h_1(r)\, |u_1'(r)|^{p-1} = a_1(r)\, f_1(u_1(r), u_2(r)), \\ (p-1)\, u_2'(r)^{p-2}\, u_2'' + \frac{N-1}{r} u_2'(r)^{p-1} + h_2(r)\, |u_2'(r)|^{p-1} = a_2(r)\, f_2(u_1(r), u_2(r)), \end{cases} \tag{2.1}$$

where we can assume in the next that $u_i'(r) \geq 0$ $(i = 1, 2)$.

First we see that radial solutions of (2.1) are any positive solutions (u_1, u_2) of the integral equations

$$\begin{cases} u_1(r) = \frac{b}{2} + \int_0^r \left(\frac{e^{-\int_0^t h_1(s)ds}}{t^{N-1}} \int_0^t s^{N-1} e^{\int_0^s h_1(s)dt} a_1(s)\, f_1(u_1(s), u_2(s))\, ds \right)^{1/(p-1)} dt, \\ u_2(r) = \frac{b}{2} + \int_0^r \left(\frac{e^{-\int_0^t h_2(s)ds}}{t^{N-1}} \int_0^t s^{N-1} e^{\int_0^s h_2(s)dt} a_2(s)\, f_2(u_1(s), u_2(s))\, ds \right)^{1/(p-1)} dt, \end{cases} \tag{2.2}$$

where $b \geq a > 0$. Our idea is to regard this as an operator equation

$$S(u_1(r), u_2(r)) = (u_1(r), u_2(r))$$

with

$$S : C[0,\infty) \times C[0,\infty) \to C[0,\infty) \times C[0,\infty)$$

defined by

$$S(u_1(r), u_2(r)) = \begin{pmatrix} \frac{b}{2} + \int_0^r \left(\frac{e^{-\int_0^t h_1(s)ds}}{t^{N-1}} \int_0^t s^{N-1} e^{\int_0^s h_1(s)dt} a_1(s)\, f_1(u_1(s), u_2(s))\, ds \right)^{1/(p-1)} dt, \\ \frac{b}{2} + \int_0^r \left(\frac{e^{-\int_0^t h_2(s)ds}}{t^{N-1}} \int_0^t s^{N-1} e^{\int_0^s h_2(s)dt} a_2(s)\, f_2(u_1(s), u_2(s))\, ds \right)^{1/(p-1)} dt \end{pmatrix}^T \tag{2.3}$$

where $u_1(0) = \frac{b}{2}$ and $u_2(0) = \frac{b}{2}$ are the central values for the system. The integration in this operator implies that a fixed point $(u_1, u_2) \in C[0,\infty) \times C[0,\infty)$ is in fact in the space $C^1[0,\infty) \times C^1[0,\infty)$. Then a solution of (2.1) will be obtained as a fixed point of the operator (2.3). To establish a solution to this operator, we use successive approximation. We define, recursively, sequences $\{u_i^k\}_{i=1,2}^{k\geq 1}$ on $[0,\infty)$ by

$$u_1^0 = u_2^0 = \frac{b}{2} \text{ for all } r \geq 0$$

and

$$\begin{aligned} \left(u_1^k(r), u_2^k(r)\right) &= S\left(u_1^{k-1}(r), u_2^{k-1}(r)\right) \\ &= \begin{pmatrix} \frac{b}{2} + \int_0^r \left(\frac{e^{-\int_0^t h_1(s)ds}}{t^{N-1}} \int_0^t s^{N-1} e^{\int_0^s h_1(s)dt} a_1(s)\, f_1\left(u_1^{k-1}(s), u_2^{k-1}(s)\right) ds \right)^{1/(p-1)} dt \\ \frac{b}{2} + \int_0^r \left(\frac{e^{-\int_0^t h_2(s)ds}}{t^{N-1}} \int_0^t s^{N-1} e^{\int_0^s h_2(s)dt} a_2(s)\, f_2\left(u_1^{k-1}(s), u_2^{k-1}(s)\right) ds \right)^{1/(p-1)} dt \end{pmatrix}^T. \end{aligned}$$

We remark that, for all $r \geq 0$, $i = 1, 2$ and $k \in N$

$$u_i^k(r) \geq \frac{b}{2},$$

and that $\left\{u_i^k\right\}_{i=1,2}^{k\geq 1}$ is an increasing sequence of nonnegative, non-decreasing functions.

We note that $\left\{u_i^k\right\}_{i=1,2}^{k\geq 1}$ satisfy

$$\left\{\begin{array}{l} (p-1)\left[\left(u_1^k\right)'\right]^{p-2}\left(u_1^k\right)'' + \left(\frac{N-1}{r}+h_1(r)\right)\left[\left(u_1^k\right)'\right]^{p-1} = a_1(r)\, f_1\left(u_1^{k-1}(r), u_2^{k-1}(r)\right), \\ (p-1)\left[\left(u_2^k\right)'\right]^{p-2}\left(u_2^k\right)'' + \left(\frac{N-1}{r}+h_1(r)\right)\left[\left(u_2^k\right)'\right]^{p-1} = a_2(r)\, f_2\left(u_1^{k-1}(r), u_2^{k-1}(r)\right). \end{array}\right. \tag{2.4}$$

Using the monotonicity of $\left\{u_i^k\right\}_{i=1,2}^{k\geq 1}$ yields

$$\begin{array}{rcl} a_1(r)\, f_1\left(u_1^{k-1}(r), u_2^{k-1}(r)\right) & \leq & a_1(r)\, f_1\left(u_1^k, u_2^k\right) \leq a_1(r) \sum\limits_{i=1}^{2} f_i\left(\sum\limits_{i=1}^{2} u_i^k, \sum\limits_{i=1}^{2} u_i^k\right), \\ a_2(r)\, f_2\left(u_1^{k-1}(r), u_2^{k-1}(r)\right) & \leq & a_2(r)\, f_2\left(u_1^k, u_2^k\right) \leq a_2(r) \sum\limits_{i=1}^{2} f_i\left(\sum\limits_{i=1}^{2} u_i^k, \sum\limits_{i=1}^{2} u_i^k\right), \end{array} \tag{2.5}$$

and, so

$$\left\{\begin{array}{l} (p-1)\left[\left(u_1^k(r)\right)'\right]^{p-1}\left(u_1^k\right)'' + \left(\frac{N-1}{r}+h_1(r)\right)\left[\left(u_1^k(r)\right)'\right]^{p} \leq a_1(r) \sum\limits_{i=1}^{2} f_i\left(\sum\limits_{i=1}^{2} u_i^k, \sum\limits_{i=1}^{2} u_i^k\right)\left(u_1^k(r)\right)', \\ (p-1)\left[\left(u_2^k(r)\right)'\right]^{p-1}\left(u_2^k\right)'' + \left(\frac{N-1}{r}+h_2(r)\right)\left[\left(u_2^k(r)\right)'\right]^{p} \leq a_2(r) \sum\limits_{i=1}^{2} f_i\left(\sum\limits_{i=1}^{2} u_i^k, \sum\limits_{i=1}^{2} u_i^k\right)\left(u_2^k(r)\right)', \end{array}\right. \tag{2.6}$$

which implies that

$$\left\{\begin{array}{l} (p-1)\left[\left(u_1^k(r)\right)'\right]^{p-1}\left(u_1^k\right)'' + \left(\frac{N-1}{r}+h_1(r)\right)\left[\left(u_1^k(r)\right)'\right]^{p} \leq a_1(r) \sum\limits_{i=1}^{2} f_i\left(\sum\limits_{i=1}^{2} u_i^k, \sum\limits_{i=1}^{2} u_i^k\right)\left(\sum\limits_{i=1}^{2} u_i^k(r)\right)', \\ (p-1)\left[\left(u_2^k(r)\right)'\right]^{p-1}\left(u_2^k\right)'' + \left(\frac{N-1}{r}+h_2(r)\right)\left[\left(u_2^k(r)\right)'\right]^{p} \leq a_2(r) \sum\limits_{i=1}^{2} f_i\left(\sum\limits_{i=1}^{2} u_i^k, \sum\limits_{i=1}^{2} u_i^k\right)\left(\sum\limits_{i=1}^{2} u_i^k(r)\right)'. \end{array}\right. \tag{2.7}$$

Let

$$a_i^R = \max\{a_i(r) : 0 \leq r \leq R\},\ i = 1,2.$$

We prove that $u_i^k(R)$ and $\left(u_i^k(R)\right)'$, both of which are nonnegative, are bounded above independent of k. Using this and the fact that $\left(u_i^k\right)' \geq 0$ $(i = 1,2)$, we note that (2.7) yields

$$\left\{\begin{array}{l} (p-1)\left[\left(u_1^k(r)\right)'\right]^{p-1}\left(u_1^k\right)'' \leq a_1^R \sum\limits_{i=1}^{2} f_i\left(\sum\limits_{i=1}^{2} u_i^k, \sum\limits_{i=1}^{2} u_i^k\right)\left(\sum\limits_{i=1}^{2} u_i^k(r)\right)' \\ (p-1)\left[\left(u_2^k(r)\right)'\right]^{p-1}\left(u_2^k\right)'' \leq a_2^R \sum\limits_{i=1}^{2} f_i\left(\sum\limits_{i=1}^{2} u_i^k, \sum\limits_{i=1}^{2} u_i^k\right)\left(\sum\limits_{i=1}^{2} u_i^k(r)\right)' \end{array}\right.$$

or, equivalently

$$\left\{\begin{array}{l} \frac{p-1}{p}\left\{\left[\left(u_1^k(r)\right)'\right]^{p}\right\}' \leq a_1^R \sum\limits_{i=1}^{2} f_i\left(\sum\limits_{i=1}^{2} u_i^k, \sum\limits_{i=1}^{2} u_i^k\right)\left(\sum\limits_{i=1}^{2} u_i^k(r)\right)', \\ \frac{p-1}{p}\left\{\left[\left(u_2^k(r)\right)'\right]^{p}\right\}' \leq a_2^R \sum\limits_{i=1}^{2} f_i\left(\sum\limits_{i=1}^{2} u_i^k, \sum\limits_{i=1}^{2} u_i^k\right)\left(\sum\limits_{i=1}^{2} u_i^k(r)\right)'. \end{array}\right.$$

But, this implies

$$\left\{\sum_{i=1}^{2}\left[\left(u_i^k(r)\right)'\right]^{p}\right\}' \leq \frac{p}{p-1}\sum_{i=1}^{2} a_i^R \sum_{i=1}^{2} f_i\left(\sum_{i=1}^{2} u_i^k(r), \sum_{i=1}^{2} u_i^k(r)\right)\left(\sum_{i=1}^{2} u_i^k(r)\right)'.$$

Integrate this equation from 0 to r. We obtain

$$\sum_{i=1}^{2}\left[\left(u_i^k(r)\right)'\right]^p \leq \frac{p}{p-1}\sum_{i=1}^{2}a_i^R\int_b^{\sum_{i=1}^{2}u_i^k(r)}\sum_{i=1}^{2}f_i(s,s)\,ds \leq \frac{p}{p-1}\sum_{i=1}^{2}a_i^R\int_0^{\sum_{i=1}^{2}u_i^k(r)}\sum_{i=1}^{2}f_i(s,s)\,ds. \tag{2.8}$$

Since $p>1$ we know that

$$(a_1+a_2)^p \leq 2^{p-1}\left(a_1^p+a_2^p\right) \tag{2.9}$$

for any non-negative constants a_i $(i=1,2)$. Using this inequality in (2.8) we have

$$2^{1-p}\left[\sum_{i=1}^{2}\left(u_i^k(r)\right)'\right]^p \leq \frac{p}{p-1}\sum_{i=1}^{2}a_i^R\int_0^{\sum_{i=1}^{2}u_i^k(r)}\sum_{i=1}^{2}f_i(s,s)\,ds,\ 0\leq r\leq R,$$

which yields

$$\left(\sum_{i=1}^{2}u_i^k(r)\right)' \leq \sqrt[p]{\frac{p2^{p-1}}{p-1}\sum_{i=1}^{2}a_i^R}\left(\int_0^{\sum_{i=1}^{2}u_i^k(r)}\sum_{i=1}^{2}f_i(s,s)\,ds\right)^{1/p},\ 0\leq r\leq R. \tag{2.10}$$

Integrating the above equation between 0 and R, we have

$$\int_b^{\sum_{i=1}^{2}u_i^k(R)}\left[\int_0^t\sum_{i=1}^{2}f_i(s,s)\,ds\right]^{-1/p}dt = I\left(\sum_{i=1}^{2}u_i^k(R)\right)-I(b) \leq \sqrt[p]{\frac{p2^{p-1}}{p-1}\left(\sum_{i=1}^{2}a_i^R\right)}R.$$

Since I is a bijection with I^{-1} increasing we obtain

$$\sum_{i=1}^{2}u_i^k(R) \leq I^{-1}\left(\sqrt[p]{\frac{p2^{p-1}}{p-1}\left(\sum_{i=1}^{2}a_i^R\right)}R+I(b)\right) \text{ for all } r\geq 0. \tag{2.11}$$

By the Keller-Osserman condition (1.2), we now conclude that $\sum_{i=1}^{2}u_i^k(R)$ is uniformly bounded above independent of k and using this fact in (2.10) shows that the same is true of $\left(\sum_{i=1}^{2}u_i^k(R)\right)'$. Thus, the sequences $u_i^k(r)$ $(i=1,2)$ are uniformly bounded above independent of k (since $r\leq R$ and $u_i^k(r)$ is non-decreasing sequence). Also, we clearly have $u_i^k(r)>0$ for all $r\geq 0$ and so our sequence is equi-continuous on $[0,R]$ for arbitrary $R>0$. Since $u_i^k(r)$ $(i=1,2)$ is a monotonic, uniformly bounded, equi-continuous sequence of functions on $[0,R]$ there exists a function $(u_1,u_2)\in C([0,R])\times C([0,R])$ such that $u_i^k(r)\to u_i^k(r)$ $(i=1,2)$ uniformly. Hence (u_1,u_2) is a fixed point of (2.3) in $C([0,R])\times C([0,R])$. Next, we extend this result to show that S has a fixed point in $C^1([0,\infty))\times C^1([0,\infty))$. Let $\left\{u_i^k(r)\right\}_{i=1,2}^{k\geq 1}$ be a sequence of fixed points defined by

$$\left(u_1^k(r),u_2^k(r)\right)=S\left(u_1^k(r),u_2^k(r)\right) \text{ on } [0,k],\ \left(u_1^k(r),u_2^k(r)\right)\in C([0,k])\times C([0,k]), \tag{2.12}$$

for $k=1,2,3,...$ As earlier, we may show that both $u_1^k(r)$ and $u_2^k(r)$ are bounded and equi-continuous on $[0,1]$. Thus by applying the Arzela-Ascoli Theorem to each sequence separately, we can derive that $\left\{\left(u_1^k(r),u_2^k(r)\right)\right\}^{k\geq 1}$ contains a convergent subsequence, $\left(u_1^{k_1^1}(r),u_2^{k_2^1}(r)\right)$, that converges uniformly on $[0,1]\times[0,1]$. Let

$$\left(u_1^{k_1^1}(r),u_2^{k_2^1}(r)\right)\to\left(u_1^1,u_2^1\right) \text{ uniformly on } [0,1]\times[0,1] \text{ as } k_1^1,k_2^1\to\infty.$$

Likewise, the subsequences $u_1^{k_1^1}(r)$ and $u_2^{k_2^1}(r)$ are each bounded and equi-continous on $[0,2]$ so there exists a subsequence $\left(u_1^{k_1^2}(r), u_2^{k_2^2}(r)\right)$ of $\left(u_1^{k_1^1}(r), u_2^{k_2^1}(r)\right)$ such that

$$\left(u_1^{k_1^2}(r), u_2^{k_2^2}(r)\right) \to \left(u_1^2, u_2^2\right) \text{ uniformly on } [0,2]\times[0,2] \text{ as } k_1^2, k_2^2 \to \infty.$$

Notice that

$$\left\{\left(u_1^{k_1^2}(r), u_2^{k_2^2}(r)\right)\right\} \subseteq \left\{\left(u_1^{k_1^1}(r), u_2^{k_2^1}(r)\right)\right\} \subseteq \left\{\left(u_1^{k}(r), u_2^{k}(r)\right)\right\}_{k\geq 2}^{\infty}$$

so $\left(u_1^2, u_2^2\right) = \left(u_1^1, u_2^1\right)$ on $[0,1]\times[0,1]$. Continuing this line of reasoning, we obtain a sequence, denoted by $\left\{\left(u_1^k(r), u_2^k(r)\right)\right\}$, such that

$$\begin{aligned}
\left(u_1^k(r), u_2^k(r)\right) &\in C([0,k])\times C([0,k]),\ k=1,2,..\\
\left(u_1^k(r), u_2^k(r)\right) &= \left(u_1^1(r), u_2^1(r)\right) \text{ for } r\in[0,1]\\
\left(u_1^k(r), u_2^k(r)\right) &= \left(u_1^2(r), u_2^2(r)\right) \text{ for } r\in[0,2]\\
&...\\
\left(u_1^k(r), u_2^k(r)\right) &= \left(u_1^{k-1}(r), u_2^{k-1}(r)\right) \text{ for } r\in[0,k-1],
\end{aligned}$$

and these functions are radially symmetric. Therefore $\left(u_1^k(r), u_2^k(r)\right)$ converges pointwise to some $(u_1(r), u_2(r))$ which satisfies

$$(u_1(r), u_2(r)) = \left(u_1^k(r), u_2^k(r)\right) \text{ if } 0\leq r\leq k,$$

Hence, $(u_1(r), u_2(r))$ is radially symmetric. Further, since $\left(u_1^k(r), u_2^k(r)\right)$ is in the form (2.12) we have that $\left(u_1^k(r), u_2^k(r)\right)$ is also equi-continuous. Pointwise convergence and equi-continuity imply uniform convergence and thus the convergence is uniform on bounded sets. Thus

$$(u_1(r), u_2(r)) \in C^1([0,\infty))\times C^1([0,\infty))$$

is a fixed point of (2.3) and a solution to (1.1) with central value $\left(\frac{b}{2}, \frac{b}{2}\right)$. Since $b\geq a>0$ was chosen arbitrarily, it follows that (1.1) has infinitely many positive entire solutions and so the first part of our theorem is proved.

The proof of i) Assume that (1.5) holds. Finally, we show that *any entire positive radial solution* (u_1, u_2) of system (1.1) is bounded. We choose $R>0$ so that

$$r^{\frac{p(N-1)}{p-1}} \sum_{j=1}^{2} e^{\frac{p}{p-1}\int_0^r h_j(t)dt} a_j(r) \text{ is non-decreasing for } r\geq R.$$

Multiplying

$$\begin{cases}
(p-1)\left[(u_1(r))'\right]^{p-1}(u_1)'' + \left(\frac{N-1}{r} + h_1(r)\right)\left[(u_1(r))'\right]^p \leq a_1(r)\sum_{i=1}^{2} f_i\left(\sum_{i=1}^{2} u_i, \sum_{i=1}^{2} u_i\right)\left(\sum_{i=1}^{2} u_i(r)\right)',\\
(p-1)\left[(u_2(r))'\right]^{p-1}(u_2)'' + \left(\frac{N-1}{r} + h_2(r)\right)\left[(u_2(r))'\right]^p \leq a_2(r)\sum_{i=1}^{2} f_i\left(\sum_{i=1}^{2} u_i, \sum_{i=1}^{2} u_i\right)\left(\sum_{i=1}^{2} u_i(r)\right)'.
\end{cases}$$

each line of this system by

$$\frac{p}{p-1} r^{\frac{p(N-1)}{p-1}} e^{\frac{p}{p-1}\int_0^r h_i(t)dt},\ (i=1,2)$$

(i represent the equation of the system that will be multiplied) and summing we have

$$\left\{r^{\frac{p(N-1)}{p-1}}\sum_{i=1}^{2}e^{\frac{p}{p-1}\int_0^r h_i(t)dt}\left[(u_i)'\right]^p\right\}' \le \frac{pr^{\frac{p(N-1)}{p-1}}}{p-1}\sum_{i=1}^{2}e^{\frac{p}{p-1}\int_0^r h_i(t)dt}a_i(r)\sum_{i=1}^{2}f_i\left(\sum_{i=1}^{2}u_i,\sum_{i=1}^{2}u_i\right)\left(\sum_{i=1}^{2}u_i\right)'.$$

and integrating gives

$$\begin{aligned}&\int_R^r\left\{s^{\frac{p(N-1)}{p-1}}\sum_{i=1}^{2}\left[e^{\frac{1}{p-1}\int_0^r h_i(t)dt}(u_i(s))'\right]^p\right\}'ds\\ \le\ &\int_R^r\frac{p}{p-1}s^{\frac{p(N-1)}{p-1}}\sum_{i=1}^{2}e^{\frac{p}{p-1}\int_0^s h_i(t)dt}a_i(s)\sum_{i=1}^{2}f_i\left(\sum_{i=1}^{2}u_i(s),\sum_{i=1}^{2}u_i(s)\right)\left(\sum_{i=1}^{2}u_i\right)'ds. \qquad (2.13)\end{aligned}$$

Hence, using (2.9) in (2.13) it gives

$$\begin{aligned}&r^{\frac{p(N-1)}{p-1}}2^{1-p}\left[\left(\sum_{i=1}^{2}e^{\frac{1}{p-1}\int_0^r h_i(t)dt}u_i(r)\right)'\right]^p-R^{\frac{p(N-1)}{p-1}}\sum_{i=1}^{2}\left[e^{\frac{1}{p-1}\int_0^r h_i(t)dt}(u_i(R))'\right]^p\\ \le\ &\int_R^r\frac{p}{p-1}s^{\frac{p(N-1)}{p-1}}\sum_{i=1}^{2}e^{\frac{p}{p-1}\int_0^s h_i(t)dt}a_i(s)\sum_{i=1}^{2}f_i\left(\sum_{i=1}^{2}u_i(s),\sum_{i=1}^{2}u_i(s)\right)\left(\sum_{i=1}^{2}u_i\right)'ds\end{aligned}$$

and thus

$$\begin{aligned}&r^{\frac{p(N-1)}{p-1}}\left[\left(\sum_{i=1}^{2}e^{\frac{1}{p-1}\int_0^r h_i(t)dt}u_i(r)\right)'\right]^p\le R^{\frac{p(N-1)}{p-1}}2^{p-1}\sum_{i=1}^{2}\left[e^{\frac{1}{p-1}\int_0^r h_i(t)dt}(u_i(R))'\right]^p+\\ &+\int_R^r\frac{p2^{p-1}s^{\frac{p(N-1)}{p-1}}}{p-1}\sum_{i=1}^{2}e^{\frac{p}{p-1}\int_0^s h_i(t)dt}a_i(s)\sum_{i=1}^{2}f_i\left(\sum_{i=1}^{2}u_i(s),\sum_{i=1}^{2}u_i(s)\right)\left(\sum_{i=1}^{2}u_i\right)'ds.\end{aligned}$$

for $r\ge R$. Noting that, by the monotonicity of $s^{\frac{p(N-1)}{p-1}}\sum_{i=1}^{2}e^{\frac{p}{p-1}\int_0^s h_i(t)dt}a_i(s)$ for $r\ge s\ge R$, we get

$$r^{\frac{p(N-1)}{p-1}}\left[\left(\sum_{i=1}^{2}e^{\frac{1}{p-1}\int_0^r h_i(t)dt}u_i(r)\right)'\right]^p\le C+\frac{p2^{p-1}}{p-1}r^{\frac{p(N-1)}{p-1}}\sum_{i=1}^{2}e^{\frac{p}{p-1}\int_0^r h_i(t)dt}a_i(r)F\left(\sum_{i=1}^{2}u_i(r)\right),$$

where

$$C=R^{\frac{p(N-1)}{p-1}}2^{p-1}\sum_{i=1}^{2}\left[e^{\frac{1}{p-1}\int_0^R h_i(t)dt}(u_i(R))'\right]^p,$$

which yields

$$\left(\sum_{i=1}^{2}e^{\frac{1}{p-1}\int_0^r h_i(t)dt}u_i\right)'\le\left[Cr^{\frac{p(1-N)}{p-1}}+\frac{p2^{p-1}}{p-1}\sum_{i=1}^{2}e^{\frac{p}{p-1}\int_0^r h_i(t)dt}a_i(r)F\left(\sum_{i=1}^{2}u_i(r)\right)\right]^{1/p}. \qquad (2.14)$$

Since $(1/p)<1$ we know that

$$(b_1+b_2)^{1/p}\le b_1^{1/p}+b_2^{1/p}$$

for any non-negative constants b_i $(i=1,2)$. Therefore, by applying this inequality in (2.14) we get

$$\begin{aligned}\left(\sum_{i=1}^{2}e^{\frac{1}{p-1}\int_0^r h_i(t)dt}u_i\right)'&\le\sqrt[p]{C}r^{(1-N)/(p-1)}+\sqrt[p]{\frac{p2^{p-1}}{p-1}\sum_{i=1}^{2}e^{\frac{p}{p-1}\int_0^r h_i(t)dt}a_i(r)}\left[F\left(\sum_{i=1}^{2}u_i(r)\right)\right]^{1/p}\\ &\le\sqrt[p]{C}r^{(1-N)/(p-1)}+\sqrt[p]{\frac{p2^{p-1}}{p-1}\sum_{i=1}^{2}e^{\frac{p}{p-1}\int_0^r h_i(t)dt}a_i(r)}\left[F\left(\sum_{i=1}^{2}e^{\frac{1}{p-1}\int_0^r h_i(t)dt}u_i(r)\right)\right]^{1/p}.\end{aligned}$$

Integrating the above inequality, we get

$$\frac{d}{dr}\int_{\sum\limits_{i=1}^{2} e^{\frac{1}{p-1}\int_0^r h_i(t)dt} u_i(R)}^{\sum\limits_{i=1}^{2} e^{\frac{1}{p-1}\int_0^r h_i(t)dt} u_i(r)} [F(t)]^{-1/p}\, dt \tag{2.15}$$
$$\leq \sqrt[p]{C} r^{(1-N)/(p-1)} \left[F\left(\sum_{i=1}^{2} e^{\frac{1}{p-1}\int_0^r h_i(t)dt} u_i(r)\right)\right]^{-1/p} + \left(\frac{p2^{p-1}}{p-1}\sum_{i=1}^{2} e^{\frac{p}{p-1}\int_0^r h_i(t)dt} a_i(r)\right)^{1/p}.$$

Integrating (2.15) and using the fact that

$$\begin{aligned}\left(\sum_{i=1}^{2} e^{\frac{p}{p-1}\int_0^r h_i(t)dt} \varphi_i(s)\right)^{1/p} &= \left(s^{p(1+\varepsilon)/2}\sum_{i=1}^{2} e^{\frac{p}{p-1}\int_0^r h_i(t)dt} a_i(s)\, s^{-p(1+\varepsilon)/2}\right)^{1/p} \\ &\leq \left(\frac{1}{2}\right)^{1/p}\left[s^{1+\varepsilon}\left(\sum_{i=1}^{2} e^{\frac{p}{p-1}\int_0^r h_i(t)dt} a_i(r)\right)^{2/p} + s^{-1-\varepsilon}\right],\end{aligned}$$

for each $\varepsilon > 0$, we have

$$\begin{aligned}&\int_{\sum\limits_{i=1}^{2} e^{\frac{1}{p-1}\int_0^r h_i(t)dt} u_i(R)}^{\sum\limits_{i=1}^{2} e^{\frac{1}{p-1}\int_0^r h_i(t)dt} u_i(r)} [F(t)]^{-1/p}\, dt \\ \leq\ & \sqrt[p]{C}\int_R^r t^{\frac{1-N}{p-1}} \left[F\left(\sum_{i=1}^{2} e^{\frac{1}{p-1}\int_0^t h_i(t)dt} u_i(t)\right)\right]^{-1/p} dt \\ &+ \left(\frac{1}{2}\right)^{1/p}\sqrt[p]{\frac{p2^{p-1}}{p-1}}\left[\int_R^r t^{1+\varepsilon}\left(\sum_{i=1}^{2} e^{\frac{p}{p-1}\int_0^t h_i(t)dt} a_i(t)\right)^{2/p} dt + \int_R^r t^{-1-\varepsilon} dt\right] \\ \leq\ & \sqrt[p]{C}\left[F\left(\sum_{i=1}^{2} e^{\frac{1}{p-1}\int_0^R h_i(t)dt} u_i(R)\right)\right]^{-1/p} \frac{p-1}{p-N} R^{\frac{p-N}{p-1}} \\ &+ \left(\frac{1}{2}\right)^{1/p}\sqrt[p]{\frac{p2^{p-1}}{p-1}}\left[\int_R^r t^{1+\varepsilon}\left(\sum_{i=1}^{2} e^{\frac{p}{p-1}\int_0^t h_i(t)dt} a_i(t)\right)^{2/p} dt + \frac{1}{\varepsilon R^{\varepsilon}}\right].\end{aligned} \tag{2.16}$$

Since the right side of this inequality is bounded (note that $u_i(t) \geq b/2$), so is the left side and hence, in light of Keller Osserman condition, the sequence $\sum\limits_{i=1}^{2} e^{\frac{1}{p-1}\int_0^r h_i(t)dt} u_i(r)$ is bounded and so $\sum\limits_{i=1}^{2} u_i(r)$ that implies finally $u_i(r)$ $(i = 1, 2)$ is a bounded function. Thus, for every $x \in \mathbb{R}^N$ $(u_1(|x|), u_2(|x|))$ is a positive bounded solution of (1.1).

The proof of ii) Suppose that a_i $(i = 1, 2)$ satisfies (1.6). Now, let (u_1, u_2) be any positive entire radial solution of (1.1) determined in the first step of the proof. Since u_i $(i = 1, 2)$ is positive for all $R > 0$ we have $u_i(R) > 0$. Since $u_i' \geq 0$, we get $u_i(r) \geq u_i(R)$ for $r \geq R$ and thus from

$$u_i(r) = u_i(0) + \int_0^r \frac{e^{-\int_0^t h_i(s)ds}}{t^{N-1}}\left(\int_0^t s^{N-1} e^{\int_0^s h_i(s)ds} a_i(s) f_i(u_1(s), u_2(s))\, ds\right)^{1/(p-1)} dt,$$

we obtain

$$\begin{cases} u_i(r) = u_i(0) + \int_0^r \left(\frac{e^{-\int_0^t h_i(s)ds}}{t^{N-1}}\int_0^t s^{N-1} e^{\int_0^t h_i(s)ds} a_i(s) f_i(u_1(s), u_2(s))\, ds\right)^{1/(p-1)} dt \ \geq u_i(R) \\ \quad + f_i^{1/(p-1)}(u_1(R), u_2(R)) \int_R^r \left(\frac{e^{-\int_0^t h_i(s)ds}}{t^{N-1}}\int_R^t s^{N-1} e^{\int_0^s h_i(s)ds} a_i(s)\, ds\right)^{1/(p-1)} dt \to \infty \text{ as } r \to \infty, \\ \text{for all } i = 1, 2.\end{cases}$$

and the proof is complete.

From the above proof and the work [3] we can easy obtain the following remarks.

Remark 2.1. *Make the same assumptions as in Theorem 1.1 on a_j, h_j and f_j excepting i)-ii). If, on the other hand, a_j satisfies*

$$\left(\frac{1}{N}\right)^{1/(p-1)} \int_0^\infty \left(e^{-\int_0^t h_j(s)ds} t a_j(t)\right)^{1/(p-1)} dt = \infty, \ j = 1, 2 \tag{2.17}$$

then system (2.2) has no nonnegative nontrivial entire bounded radial solution on $\mathbb{R}^N$.

Remark 2.2. *(see and [3] for the proof) Make the same assumptions as in Theorem 1.1 on a_j, h_j and f_j excepting i)-ii). If (2.2) has a nonnegative entire large solution, then a_j $(j = 1, 2)$ satisfy*

$$\int_0^\infty r^{1+\varepsilon} \left(\sum_{j=1}^{2} e^{\frac{p}{p-1}\int_0^t h_j(t)dt} a_j(t)\right)^{2/p} dr = \infty, \tag{2.18}$$

for every $\varepsilon > 0$.

Remark 2.3. *As we have observed in the article [3] the above proofs can be adopted to obtain the same results for systems with indefinite number of equations.*

References

[1] C. Bandle and M. *Marcus, Large solutions of semilinear elliptic equations: existence, uniqueness and asymptotic behavior*, Journal d'Analyse Mathématique 58, Pages 9-24, 1992.

[2] D.-P. Covei, *Large and Entire Large Solution for a Quasilinear Problem*, Nonlinear Analysis: Theory, Methods & Applications, Volume 70, Issue 4, Pages 1738-1745, 15 February 2009.

[3] D.-P. Covei, *Radial and nonradial solutions for a semilinear elliptic system of Schrödinger type*, Funkcialaj Ekvacioj-Serio Internacia, accepted manuscript on December, 08, 2010. (to appear in Funkcialaj Ekvacioj FE., Vol. 54, No.3, 2011.) http://arxiv.org/PS_cache/arxiv/pdf/1105/1105.2652v1.pdf

[4] J. B. Keller, *On solution of* $\Delta u = f(u)$, Communications on Pure and Applied Mathematics, 10 (1957), 503-510.

[5] J. B. Keller, *Electrohydrodynamics I. The Equilibrium of a Charged Gas in a Container*, Journal of Rational Mechanics and Analysis, Volume 5, Nomber 4, 1956.

[6] K.-S. Cheng and W.-M. Ni, *On the structure of the conformal scalar curvature equation on R^{N}*, Indiana University Mathematics Journal, Volume 41, No. 1, 1992.

[7] J. Matero, *Quasilinear elliptic equations with boundary blow-up*, Journal D'Analyse Mathèmatique, Volume 69, Pages 229-247, 1996.

[8] J. Matero, *Nonlinear elliptic problems with boundary blow-up*, Uppsala Dissertations in Mathematics, 1997.

[9] W.-M. Ni, *On the elliptic equation $\Delta u + K(x)u^{(N+2)/(N-2)} = 0$, its generalizations, and applications in geometry*, Indiana University Mathematics Journal, Volume 31, Nomber 4, 1982.

[10] R. Osserman, *On the inequality $\Delta u \geq f(u)$*, Pacific Journal of Mathematics 7, Pages 1641-1647, 1957.

[11] J. D. Peterson and A. W. Wood, *Large solutions to non-monotone semilinear elliptic systems*, Journal of Mathematical Analysis and Applications, In Press, Corrected Proof, Available online 31 May 2011.

[12] X. Zhang and L. Liu, *The existence and nonexistence of entire positive solutions of semilinear elliptic systems with gradient term*, Journal of Mathematical Analysis and Applications, Volume 371, Issue 1, 1 November 2010, Pages 300-308.